4

Catching 'em Off Guard: Mixed Strategy Nash Equilibria

Float like a butterfly, sting like a bee.
Mohammed Ali

4.1 Introduction

Have you ever wondered why so many of God's creatures are bilaterally symmetric? Well, one reason is that most such creatures must chase and/or flee predators. The fox, for instance, is faster than the rabbit, but the rabbit can change direction more easily than the fox. Suppose the rabbit were strong on the left side and weak on the right side. Then it would easily jump to the right, but with difficulty to the left. Thus it would mostly jump to the right. Knowing that the rabbit will jump the right, the fox will trail the rabbit with a bias to the right, and with a predisposition to jump right, thus making the benefit of jumping right less valuable to the rabbit. But then mutant rabbits that reversed this pattern of strength and weakness would more likely survive, driving out the existing rabbits. And so on. In equilibrium, the rabbit would be bilaterally symmetric, as would the fox. So mixed strategies, limiting an opponent's options by mixing up one's actions, are of fundamental importance in strategic interaction.

Even though defined as though played a single time, games with mixed strategy equilibria can rarely be properly understood except in terms of repeated interactions. A few games, such as *Throwing Fingers* (§4.10), *Tennis Strategy* (§4.17), and *One-Card Two-Round Poker* (§4.11) can be thought of as the same agents playing the game repeatedly. But most are best thought of by considering a large population of agents who are randomly paired to play the game in each of many periods. In equilibrium, each agent then plays a pure strategy, but the fraction of

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the population playing each pure strategy is that given by the mixed strategy Nash equilibrium. While such a population model is formally the subject matter of evolutionary dynamics (Chapter 9), the intuition is developed in many of the models in this chapter.

After the introductory definitions, algorithms, and examples, *One-Card Two-Round Poker with Bluffing* is an excellent problem to present in class, explaining first the analytical derivation of the equilibrium, and then describing the artificial life simulation, which shows that low-level intelligence, in an evolutionary setting, can attain a sophisticated mixed strategy equilibrium that in classical game theory could only be justified by assuming highly sophisticated, 'rational' players. A second beautiful example of population thinking is *Orange-Throat, Blue-Throat and Yellow-Striped Lizards* (§4.15), in which males of the lizard species *Uta stansburiana* have been shown to play Rock, Paper, and Scissors in their mating behavior. Another fine classroom example is *Sex Ratios as Nash Equilibria* (§4.16), which explains a deep regularity of nature that many students will never even have thought about—equal sex ratios—by showing that it is a mixed strategy equilibrium of an appropriately defined game. An instructor who prefers to say closer to the economy could substitute *Mutual Monitoring* and *Cooperation in a Partnership* (§4.30) and the follow section, which nicely illustrate the interplay of cooperation and conflict in teams.

§4.44 develops the equivalence of mixed and behavioral strategies in extensive form games. I have found that students consider the equivalence intuitively obvious, although this cannot be the case, since it is true only assuming perfect recall. Theoretically-oriented students should be urged to work through this section, but it can be skipped until behavioral strategies make a serious appearance in later chapters. For most purposes the summary in §4.12 will suffice.

4.2 Mixed Strategies: Basic Definitions

Suppose a player has pure strategies $s_1, \ldots, s_k$ in a normal form game. A *mixed strategy* for the player is a probability distribution over $s_1, \ldots, s_k$; i.e., a mixed strategy has the form

$$\sigma = p_1 s_1 + \ldots + p_k s_k,$$

where $p_1, \ldots p_k$ are all nonnegative and $\sum_1^n p_j = 1$. By this we mean that the player chooses s_j with probability p_j, for $j = 1, \ldots, k$. We call p_j the *weight* of s_j in σ. If all the p_j's are zero except one, say $p_l = 1$,

we say σ is a *pure strategy*, and we write $\sigma = s_l$. We say that pure strategy s_j is *used* in mixed strategy σ if $p_j > 0$. We say a strategy is *strictly mixed* if it is not pure, and we say that it is *completely mixed* if all pure strategies are used in it. We call the set of pure strategies used in a mixed strategy σ_i the *support* of σ_i.

In an n-player normal form game where player i has pure strategy set S_i for $i = 1, \ldots, n$, a *mixed strategy profile* $\sigma = (\sigma_1, \ldots, \sigma_n)$ is the choice of a mixed strategy σ_i by each player. We define the *payoffs* to σ as follows. Let $\pi_i(s_1, \ldots, s_n)$ be the payoff to player i when players use the pure strategy profile $(s_1, \ldots, s_n)$, and if s is a pure strategy for player i, let p_s be the weight of s in σ_i. Then we define

$$\pi_i(\sigma) = \sum_{s_1 \in S_1} \cdots \sum_{s_n \in S_n} p_{s_1} p_{s_2} \ldots p_{s_n} \pi_i(s_1, \ldots, s_n).$$

This is a formidable expression, but the idea behind it is simple. We assume the players' choices are made independently, so the probability that the particular pure strategies $s_1 \in S_1, \ldots, s_n \in S_n$ will be used is simply the product $p_{s_1} \ldots p_{s_n}$ of their weights, and the payoff to player i in this case is just $\pi_i(s_1, \ldots, s_n)$. We get the expected payoff by multiplying and adding up over all n-tuples of mixed strategies.

We say that strategy profile $(\sigma_1^*, \ldots, \sigma_n^*)$ is a *mixed strategy Nash equilibrium* if, for each player $i = 1, \ldots, n$,

$$\pi_i(\sigma^*) \geq \pi_i(\sigma_i, \sigma_{-i}^*),$$

where σ_i is any mixed strategy for player i, and the σ_{-i} notation is as defined in §1.5. Notice that this is exactly the same as in the case of pure strategies, except we use σ's instead of s's.

4.3 The Fundamental Theorem of Mixed Strategy Nash Equilibrium

John Nash showed that every finite game has a Nash equilibrium in mixed strategies (Nash 1950). The following *Fundamental Theorem* develops the principles for finding Nash equilibria.

Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a mixed strategy profile for an n-player game. For any player $i = 1, \ldots, n$, let σ_{-i} represent the mixed strategies used by all the players other than player i. The *Fundamental Theorem of Mixed Strategy Nash Equilibrium* says that σ is a Nash equilibrium if and only if, for any player $i = 1, \ldots, n$ with pure strategy set S_i,

(a) if $s, s' \in S_i$ occur with positive probability in σ_i, then the payoffs to s and s', when played against σ_{-i}, are equal;

(b) if s occurs with positive probability in σ_i and s' occurs with zero probability in σ_i, then the payoff to s' is less than or equal to the payoff to s, when played against σ_{-i}.

The proof of the Fundamental Theorem is straightforward: Suppose σ is the player's mixed strategy in a Nash equilibrium that uses s with probability $p > 0$ and s' with probability $p' > 0$. If s has a higher payoff than s' when played against σ_{-i}, then i's mixed strategy that uses s with probability $(p + p')$, does not use s', and assigns the same probabilities to the other pure strategies as does σ, has a higher payoff than σ, so σ is not a best response to σ_{-i}. This is a contradiction, which proves the assertion. The rest of the proof is similar.

4.4 Solving for Mixed Strategy Nash Equilibria

	U	D
L	a_1, a_2	b_1, b_2
R	c_1, c_2	d_1, d_2

This is an example of the general method of finding mixed strategy equilibria in normal form games. Consider the game at the right. First, of course, you should check for pure strategy equilibria, as *per* the previous chapter. To check for a completely mixed strategy equilibrium, we use the Fundamental Theorem (4.3). Suppose the column player uses the strategy $\sigma = \alpha U + (1 - \alpha)D$ (i.e., plays U with probability α). Then if the row player uses both L and R, they must both have the same payoff against σ. The payoff to L against σ is $\alpha a_1 + (1 - \alpha)b_1$, and the payoff to R against σ is $\alpha c_1 + (1 - \alpha)d_1$. Equating these two, we find

$$\alpha = \frac{d_1 - b_1}{d_1 - b_1 + a_1 - c_1}.$$

For this to make sense, the denominator must be non-zero, and the right hand side must lie between zero and one. Note that *column* player's strategy is determined by the requirement that *row* player's two strategies be equal.

Now suppose the row player uses strategy $\tau = \beta L + (1-\beta)R$ (i.e., plays L with probability β). Then if the column player uses both U and D, they must both have the same payoff against τ. The payoff to U against

τ is $\beta a_2 + (1-\beta)c_2$, and the payoff to D against τ is $\beta b_2 + (1-\beta)d_2$. Equating these two, we find

$$\beta = \frac{d_2 - c_2}{d_2 - c_2 + a_2 - b_2}.$$

Again, for this to make sense, the denominator must be non-zero, and the right hand side must lie between zero and one. Note that now *row* player's strategy is determined by the requirement that *column* player's two strategies are equal.

(a) Suppose the above really is a mixed strategy equilibrium. What are the payoffs to the two players?

(b) Note that to solve a 2×2 game, we have checked for five different 'configurations' of Nash equilibria—four pure and one mixed. But there are four more possible configurations, in which one player uses a pure strategy and the second player uses a mixed strategy. Show that if there is a Nash equilibrium in which the row player uses a pure strategy (say L) and the column player uses a completely mixed strategy, then *any* strategy for the column player is a best response to L.

(c) How many different configurations are there to check for in a 2×3 game? In a 3×3 game?

(d) Can you generalize to the number of possible configurations of Nash equilibria in an $n \times m$ normal form game?

4.5 Reviewing the Terminology

(a) Define a *mixed strategy*, and write the expression for the payoff to using a mixed strategy as a function of the payoffs to the underlying pure strategies.

(b) Write the condition for a set of mixed strategies to form a Nash equilibrium.

(c) We say a Nash equilibrium is *strict* if the strategy used by each player in this equilibrium is the only best response to the strategies used by the other player. Define a *strictly mixed* strategy, and show that if any player in a Nash equilibrium uses a strictly mixed strategy, then the equilibrium is not strict.

4.6 Big Monkey and Little Monkey Revisited

Find the mixed strategy Nash equilibrium to the simultaneous-move Big Monkey and Little Monkey game discussed at the end of §1.2.

4.7 Dominance Revisted

Show that if a game has a solution by the iterated elimination of strictly dominated strategies (§2.1), then this solution is the only Nash equilibrium of the game. Hint: use the Fundamental Theorem to show that each strictly dominated strategy has weight zero in a mixed strategy Nash equilibrium.

4.8 Competition on Main Street Revisited

In *Competition on Main Street* (§3.2) you showed that there is no pure strategy equilibrium with three agents. Suppose that general stores can only be set up at locations $0, 1/n, \ldots, (n-1)/n, 1$ (multiple stores can occupy the same location).

(a) Let $\pi(x, y, z)$ be the payoff to the agent choosing location x when the other two agents choose y and z. Find an expression for $\pi(x, y, z)$.

(b) Show that for $n = 4$ there is a mixed strategy Nash equilibrium in which each agent locates at points $1/4$ and $3/4$ with probability $1/7$ and point $1/2$ with probability $5/7$;

(c) Show that for $n = 6$ there is a mixed strategy Nash equilibrium in which each agent locates at points $1/3$, $1/2$ and $2/3$ each with probability $1/3$.

(d)* Show that for $n = 10$ there is no mixed strategy equilibrium in which all agents locate within one location of the center, but there is one in which they locate within two locations of the center. Show that locating at $3/10$, $2/5$, $1/2$, $4/5$ and $7/10$ with equal probability is such an equilibrium.

(e)* If you have the appropriate mathematical software (e.g., Mathematica or Maple), or if you have a long weekend with nothing to do, find mixed strategy equilibria for $n = 12, 14, 16$.

4.9 Battle of the Sexes

Elisabetta and Alfredo love each other so much that they would rather be together than be apart. But Alfredo wants to go gambling, and Elisabetta wants to go to the opera. Their payoffs are described in the insert. Show there are two pure strategy equilibria and one mixed strategy equilibrium for this game. Show that Alfredo and Elisabetta would be better off if they coordinated their activities.

Alfredo \ Elisabetta	g	o
g	2,1	0,0
o	0,0	1,2

4.10 Throwing Fingers

Two people play a game. Both throw down one hand with 1 or 2 fingers out. If the sum is even, player 1 wins \$1 from player 2. Otherwise player 2 wins \$1 from player 2.

(a) Write the normal form matrix for this game, and show that there are no pure strategy equilibria;

(b) Show that each player has a mixed strategy equilibrium in which 1 finger is played with probability 1/2.

4.11 One-card, Two-round Poker with Bluffing

Two card players start by each putting \$2 into the 'pot.' Player 1 is dealt a card, which with equal probability is either H (High) or L (Low). After looking at his card (which Player 2 cannot see), he either raises or folds. If he folds, the game is over and Player 2 takes the pot. If he raises, he must put an additional \$2 into the pot, and player 2 must now either stay or fold. If Player 2 folds, the game is over and he loses the pot. If he stays, he must put an additional \$2 into the pot to meet the bet of Player 1, and Player 1 has another turn. Player 1 must again raise or fold. If he folds, he loses the pot, and if he plays, he must put another \$2 into the pot, and Player 2 has a final turn. Player 2 now must either fold or stay. If Player 2 folds, he loses the pot and the game is over. If he stays, he must put an additional \$2 into the pot, and Player 1 must show his card. If it is H, he wins the pot, and if it is L, he loses the pot.

These rules sound complicated, but they are really simple: both players start out by *ante*'ing \$2. On Player 1's turn he must raise or fold, and on Player 2's turn, he must see or fold. Each player has two turns. The extensive form for this game is shown in Figure 4.1.

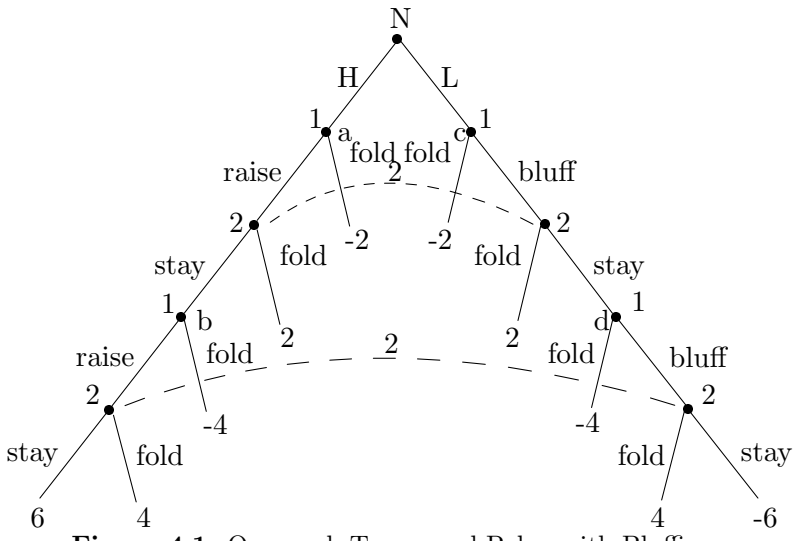


Figure 4.1: One-card, Two-round Poker with Bluffing

It is easy to see that Player 2 has three pure strategies: ss (stay,stay), sf (stay,fold), and f (fold). Player 1 has nine strategies: rrbb (raise,raise on H, and bluff,bluff on L), rrbf (raise,raise on H, and bluff,fold on L), rrf (raise,raise on H, and fold on L), rfbb (raise,fold on H, and bluff,bluff on L), rfbf (raise,fold on H, bluff,fold on L), fbb (fold on H, bluff,bluff on L), fbf (fold on H, bluff,fold on L), rff (raise,fold on H, fold on L), and ff (fold on H, fold on L).

(a) Show that Player 1 has only 3 undominated pure strategies, and find the optimal mixed strategy.

(b) Find the optimal mixed strategy for player 2.

4.11.1 Simulating One-card Two-round Bluff

We can simulate One-Card Two-Round Bluff by creating in the computer *artificial life* creatures with very little information processing capacity (none, in fact). The creature's genome consists of a mixed strategy (i.e., a probability distribution over the three non-dominated strategies) for Player 1, and similarly for Player 2. In this simulation, I created 200 players of each type, and assigned them mixed strategies randomly. In each round of play, partners are randomly assigned, and every 100 rounds, we allow reproduction to take place. In this simulation, reproduction consisted in killing off the player with the lowest score, and allowing

the player with the highest score to reproduce, with mutation in the genome at rate 20%. The simulation ran for 50,000 rounds. The results of one run of the simulations for the distribution of Player 2 types in the economy are shown in Figure 4.2. Note that after 25,000 rounds, the frequency of each strategy has settled down to the theoretically predicted equilibrium value.

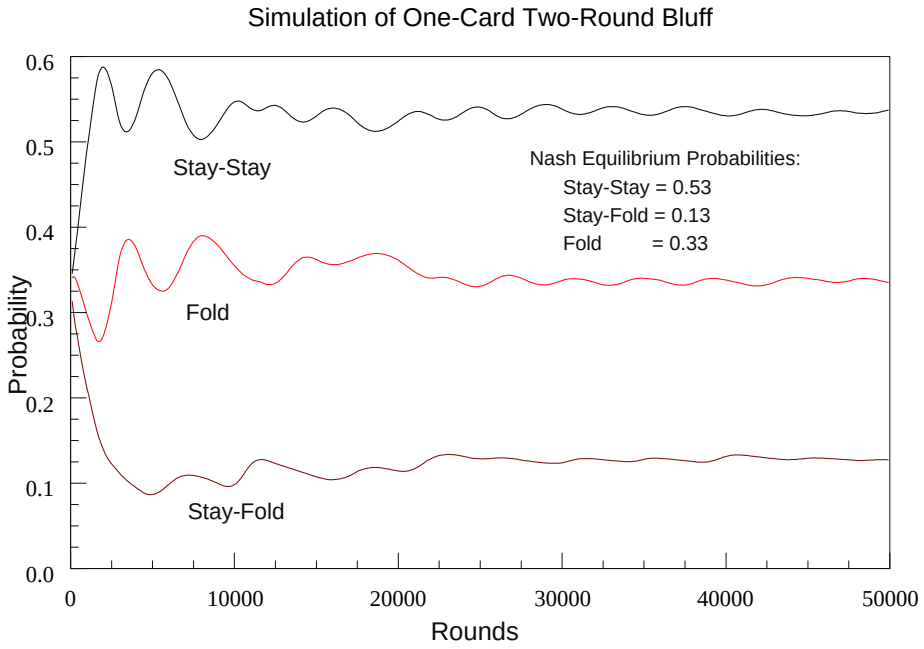


Figure 4.2: Simulation of One-Card Two-Round Bluff

4.12 Behavioral Strategies in Extensive Form Games

While the concept of a mixed strategy is very natural in a normal form game, the most natural definition of randomizing in an extensive form game is that the *behavioral strategy*. A behavioral strategy for a player in an extensive form games is a probability distribution over the actions available at each point in the game where the player moves.

Consider, for example, One-Card Two-Round Bluff (§4.11). If you solved this problem correctly, you found that player 1 uses the mixed strategy $(8/15)rrbb + (2/15)rrbf + (1/3)rrf$. Referring to Figure 4.1,

notice that all three pure strategies involve raising at nodes a and b (since $rrbb$, $rrbf$ and rrf all start with rr). At node c, $rrbb$ and $rrbf$ involve raising and rrf involves dropping so the player raises with probability $8/15 + 2/15 = 2/3$ at c. If we get to node d, player 1 has used either $rrbb$ or $rrbf$, and the probability of having used $rrbb$ is the conditional probability (§16.2)

$$\text{P[raise on both rounds|raise on round one]} = \frac{\text{P[raise on both rounds]}}{\text{P[raise on round one]}} = \frac{(8/15)}{8/15 + 2/15} = 4/5,$$

which is the probability of raising at node d. We conclude that player 1 should "raise with probability 1 at a and b, raise with probability $2/3$ at c, and raise with probability $4/5$ at d." This defines an Nash behavioral strategy for player 1.

Formally, p_i is a behavioral strategy for player i in extensive form game $\mathcal{G}$ if for every information set ν of player i, if $a^\nu = \{a_1^\nu, \ldots, a_{k_\nu}^\nu\}$ is the set of actions associated with ν, then p_i is a probability distribution over a^ν.[1]

A *behavioral strategy profile* $p = (p_1, \ldots, p_n)$ for $\mathcal{G}$ is the choice of a behavioral strategy p_i for each player i. To simplify the notation, we shall also include in a behavioral strategy profile p the probability distributions over the nodes where Nature moves in $\mathcal{G}$ (we assume all branches where Nature moves have strictly positive probability). Thus a behavioral strategy profile p determines, for every non-terminal node a of $\mathcal{G}$, a probability distribution $p(a^\nu)$ for the actions a^ν at where $p(a^\nu) = p_i(a^\nu)$ if $a \in \nu$ for some information set ν where player i chooses.

The *payoffs to a behavioral strategy profile* p are the expected payoffs to the players, assuming they use the probabilities given by p. We specify this as follows. For any two nodes a and a' in the game tree, there is either no path from a to a', or there is a unique path $(b_1, \ldots, b_k)$ from a to a'. In the former case we write $\text{P}[a, a'|p] = 0$, and in the latter case we write

$$\text{P}[a, a'|p] = \prod_{i=1}^{k} p(b_i), \tag{4.1}$$

which is the probability of moving from a to a', given behavioral strategy p. Notice that since a branch is an action, it makes sense to write $p(b_i)$.

[1] We should really say that the *restriction* of p_i to a^ν is a probability distribution over a^ν, since p_i is in fact a probability distribution over *each* information set at which i moves.

Now suppose r is the root node of $\mathcal{G}$, T is the set of terminal nodes of $\mathcal{G}$, and the payoff to player i at node $t \in T$ is $\pi_i(t)$. The payoff $\pi_i(p)$ to player i, given behavioral strategy profile p, is then

$$\pi_i(p) = \sum_{t \in T} \mathrm{P}[r, t|p]\pi_i(t).$$

There is a natural equivalence between behavioral strategies of extensive form games and mixed strategies of their normal form counterparts.[2] The following assertions, which are developed formally in §4.44, are sufficient for most purposes: (a) Suppose we are given a behavioral strategy p_i for player i. Then a pure strategy $s_i \in S_i$ for player i picks out one action from each of i's information sets. Let $p_i(s_i)$ be the product of the probabilities of each of these actions, given by p_i, and let $\sigma_i = \sum_{s_i \in S_i} p_i s_i$. Then σ_i is the mixed strategy for i corresponding to p_i; (b) given mixed strategy σ_i and an information set ν belonging to player i, let $S_i[\nu]$ be i's pure strategies that reach ν with positive probability for some choice of strategies of the other players, and let q be the sum of the weights of the pure strategies in $S_i[\nu]$ as they occur in σ_i. Also for each action $a_j^\nu \in \nu$, let q_j be the sum of the weights of the subset of $S_i[\nu]$ that choose a_j^ν. Finally, let $p_i(a_j^\nu) = q_j/q$, provided $q \neq 0$ (if $q = 0$, define p to be an arbitrary probability distribution on ν). Then p_i is the behavioral strategy corresponding to σ_i; (c) if σ is the mixed strategy representation of behavioral strategy p, then p is a behavioral strategy representation of σ; (d) if p is a behavioral strategy profile with mixed strategy representation σ, then the payoffs to p and to σ are the same; (e) if σ is a Nash equilibrium of the normal form representation of the extensive game $\mathcal{G}$, and if p is a behavioral strategy representation of σ, then at every information set ν, p_ν is a best response to $p_{-\nu}$, which we define to be the probability distributions over all information sets except ν, given by p.

4.13 Lions and Antelope

Two lions out hunting see Big Antelope and Little Antelope in the distance. They will surely catch whatever prey they chase, whether alone or together. However if they pick different antelopes to chase, there is no need to share, while if they go after the same antelope, each will get

[2]This assumes perfect recall, but if using mixed strategies makes sense in an extensive form game, the game must have perfect recall, since a player can refer to the pure strategy he is using to distinguish between earlier choice nodes later in the game.

only half of the kill, and the other antelope will escape. Suppose their decisions are independent, the caloric value of Big Antelope is c_b, the caloric value of Little Antelope is c_l, and $0 < c_l < c_b$.

(a) Write a normal form matrix for the game, and determine the conditions (inequality relations on c_l and c_b) under which there is at least one pure strategy equilibrium.

(b) Show that if it does not pay for them both to chase Big Antelope, then there is a mixed strategy equilibrium. Find the probability that each lion chases Big Antelope in this equilibrium.

(c) Can the payoff to the mixed strategy equilibrium exceed the payoff to catching Little Antelope in a pure strategy equilibrium? Can it fall short of the pure strategy payoff?

4.14 The Santa Fé Bar

In Santa Fé there is nothing to do at night but look at the stars or go to the local bar. Let us define the utility of looking at the stars as 0, and let the cost of walking over to the bar be 1. Suppose the utility from being at the bar is 2 if there are fewer than three people at the bar, and 1/2 if there are three or more people at the bar.

Suppose there are three people in Santa Fé.

(a) How many pure strategy equilibria are there? What is the total social surplus from a pure strategy equilibrium?

(b) Show that there is a unique symmetric mixed strategy equilibrium in which each resident goes to the bar with probability $\sqrt{6}/3 \approx 81.65\%$. Show that the total social surplus is now zero—the bar might as well not exist!

(c) Now suppose for each player $i = 1, 2, 3$ the cost of walking to the bar is c_i, the payoff when there are fewer than three people in the bar is a_i, and the payoff otherwise is b_i, where $0 < b_i < c_i < a_i$. Show that if there is any mixed strategy equilibrium, it is unique, and find the conditions under which it exists. Show that the bar might as well not exist.

(d) Generalize the problem to n people and a more general payoff structure. Show that the bar still might as well not exist.

4.15 Orange-Throat, Blue-Throat and Yellow-Striped Lizards

The side-blotched lizard *Uta stansburiana* has three distinct male types: orange-throats, blue-throats, and yellow-striped. The orange-throats are violently aggressive, keep large harems of females (up to seven), and defend large territories. The blue-throats are less aggressive, keep small harems (usually three females), and defend small territories. The yellow-sides are very docile but they look like females, so they can infiltrate another male's territory and secretly copulate with the females. Field researchers note that there is regular succession from generation to generation, in which orange-throated males are a majority in one period, followed by a majority of yellow-sided males, who are followed in turn by a majority of blue-throated males, and finally by a new majority of orange-throated males, thus completing the cycle.

	Orange Throat	Blue Throat	Yellow Striped
Orange Throat	0,0	-1,1	1,-1
Blue Throat	1,-1	0,0	-1,1
Yellow Striped	-1,1	1,-1	0,0

This cycle occurs because the orange-throats have so large a territory and so large a harem that they cannot guard effectively against the sneaky yellow-sided males, who thus secure a majority of the copulations. When the latter are very common, however, the males of the blue-throated variety benefit, since they can detect and eject the yellow-sides, as the blue-throats have smaller territories and fewer females to monitor. However when the blue-throats predominate, the vigorous orange-throats do comparatively well, since they can chase the blue-throats and acquire their harems and territories.

The game underlying this is the familiar children's game *Rock, Paper, and Scissors*, with the accompanying payoff structure. Notice that just as in the lizard case, each 'type' (rock, paper, scissors), receives zero payoff playing against itself, but is superior to one of its two dissimilar adversaries and is inferior to the other of its dissimilar adversaries (yellow-striped beats orange-throat but is beaten by blue-throat; orange-throat beats blue-throat but is beaten by yellow-striped; blue-throat beats yellow-striped but is beaten by orange-throat). Show that the only Nash equilibrium to this game is the mixed strategy equilibrium in which each strategy is played with equal probability.

After you have learned how to model game dynamics, we will return to this problem and show that under a replicator dynamic, the male lizard population does indeed cycle among the three forms in successive breeding periods (§9.11).

4.16 Sex Ratios as Nash Equilibria

Most organisms that employ sexual reproduction have two sexes: male and female. The fraction of a female's offspring that are female is determined by genetic factors, and hence is heritable. In many species (e.g., most animals), the fraction is almost exactly 1/2, even if the viability of males and females (i.e., the probability that they mature to the point of sexual reproduction) are very different. Why is this the case?

To streamline the process of solving this problem, suppose all females breed simultaneously, and their offspring comprise the next generation of birds (i.e., birds live for only one breeding period). Unless otherwise stated, you should assume (i) females 'choose' a ratio u of sons to daughters born that maximizes the expected number of their genes among their grandchildren; (ii) each female produces c offspring; (iii) males and females contribute an equal number of genes to their offspring; (iv) all males are equally likely to sire an offspring; (v) there is random mating in the next generation; and (vi) the next generation is so large that no single female can affect the ratio v of males to females in the next generation.

(a) Show that $u = 1/2$ in equilibrium; i.e., a female produces half sons and half daughters.

Hint: Call the birds surviving to maturity in the next generation the 'breeding pool.' Let s and d be the number of sons and daughters in the breeding pool. Explain why $\alpha = d/s$ is the expected number of offspring of a male in the breeding pool. Find an expression for α in terms of σ_m and σ_f, the fraction of sons and daughters, respectively, who survive to maturity, and v. Now write an expression for $f(u, v)$, the number of grandchildren of a female, in terms of α and the other parameters of the problem (u, v, s_f, s_m, and c). Choose u to maximize $f(u, v)$, and show the only Nash equilibrium is when $u = v$.

(b) Suppose there are n females, and n is sufficiently small that a single female's choice *does* affect the ratio of daughters to sons. Show that an equal number of daughters and sons remains a Nash equilibrium.

(c) Suppose that instead of only breeding once, a fraction δ_m of breeding males and δ_f of breeding females die in each period, and the rest remain in the mating pool. Find an expression for the equilibrium

ratio of males to females, and show that this ratio remains 1/2 if $\delta_f = \delta_m$.

(d) Suppose the species is haplo-diploid (many bee species are). This means that males have only one copy of each gene, which they get from their mother (i.e., males come from unfertilized eggs), while females have two copies of each gene, one from each parent. Find the equilibrium ratio of daughters to sons assuming birds live for one breeding period, and females maximize the number of copies of their genes in their grandchildren.

4.17 Tennis Strategy

In tennis, the server can serve either to the receiver's backhand or the receiver's forehand. The receiver can anticipate that the ball will come either to her forehand or her backhand. If the receiver anticipates correctly, she is more likely to return the ball. On the other hand, the server has a stronger backhand than forehand serve. Therefore the receiver will return a correctly anticipated backhand serve with 60% probability and a correctly anticipated forehand serve with 90% probability. If the receiver wrongly anticipates a forehand, her return is good 20% of the time, while if she wrongly anticipates a backhand, her return is good 30% of the time.

(a) Write the normal form matrix for this game, and show that there are no pure strategy equilibria.

(b) Find a mixed strategy equilibrium for this game. What fraction of the time does the receiver return the serve successfully?

(c) Show that if the receiver's backhand gets a little better, so a correctly anticipated backhand is returned 65% of the time instead of 60%, then the server should go to the forehand more frequently and the receiver should anticipate a forehand more frequently (find the exact rates!). Show that the receiver now returns the serve successfully a greater percent of the time.

(d) The moral of the story: when you improve a shot, you get to use it less, but you win more frequently.

4.18 A Mating Game

Consider a mating system in which there are males and females, 50% of each sex being *hierarchical* (H) and the other half *egalitarian* (E). When a male meets a female to mate, their sex is visible, but neither knows the others's H/E type. There are two mating strategies: *forward* (F) and *reserved* (R). Females prefer their partners to be reserved, but males prefer to be forward. In addition, when a pair of hierarchicals meet, they both prefer that one be forward and the other reserved, but when a pair of egalitarians meet, they both prefer to play the same strategy—both forward or both reserved. The payoffs are depicted in Figure 4.3.

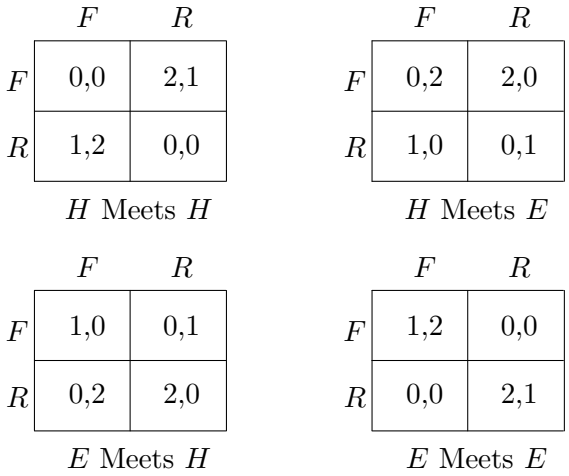


Figure 4.3: Mating Game Payoffs, where the Female is the row player.

There are four pure strategies: FF (forward if H, forward if E), FR (forward if H, reserved if E), RF (reserved if H, forward if E), RR (reserved if H, reserved if E).

(a) Explain why the payoffs in Figure 4.3, where the female is the row player, reflect the preferences of the players;

(b) Show that there is one pure strategy Nash equilibrium, given by RF for females and FF for males, and all players have expected payoff equal to 1;

(c) Show that there is a mixed strategy equilibrium in which males play FF with probability 1/3 and RF with probability 2/3, and females play any combination of strategies leading them to be forward with

probability 1/3; find the payoffs and show that this equilibrium is Pareto-dominated (§3.1) by the pure strategy equilibrium;

(d) Show that there is a 'mirror' mixed strategy equilibrium in which males play FF with probability 1/3 and FR with probability 2/3, and females play any combination of strategies leading them to be forward with probability 2/3; find the payoffs and show that this equilibrium is Pareto-dominated (§3.1) by the pure strategy equilibrium.

4.19 Hard Love

A mother wants to help her unemployed son financially, provided he is looking for work. The son searches for work only if he can't depend on his mother for support, and may not find work even if he searches. The payoff matrix is as shown in insert. Show that there are no pure strategy Nash equilibria, and find the mixed strategy equilibrium.

		Son
Mom	Seek Work	Watch Soaps
Help Son	3,2	-1,3
Hard Love	-1,1	0,0

4.20 Coordination Failure I

Determine the unique mixed Nash equilibrium of the normal form game in the insert. Show that if either player adopts any strategy other than his unique Nash strategy, the optimal response by the other player will result in a superior outcome for both. In this case, then, the Nash equilibrium is the worst of all possible worlds (note that in the Prisoner's Dilemma, the Nash solution is not a Pareto minimum, since a player does worse when his partner defects and he cooperates.)

0,0	50,40	40,50
40,50	0,0	50,40
50,40	40,50	0,0

4.21 Advertising Game

Three firms (players 1, 2, and 3) put three items on the market, and can advertise these products either on morning TV or evening TV. A firm advertises exactly once per day. If more than one firm advertises at the same time, their profits are zero. If exactly one firm advertises

in the morning, its profit is 1, and if exactly one firm advertises in the evening, its profit is 2. Firms must make their daily advertising decisions simultaneously.

Show that the Nash equilibria take the following form:

(a) One firm always chooses morning, another always chooses evening, and the third chooses morning with any probability.

(b) All firms advertise in the morning with probability $\sqrt{2} - 1$.

4.22 Preservation of Ecology Game

Each of three firms (1,2, and 3) uses water from a lake for production purposes. Each has two pure strategies: purify sewage (strategy 1) or divert it back into the lake (strategy 2). We assume that if zero or one firm diverts its sewage into the lake, the water remains pure, but if two or more firms do, the water is impure and each firm suffers a loss of 3. The cost of purification is 1.

Show that the Nash equilibria can be characterized by:

(a) One firm always pollutes and the other two never do;

(b) All firms always pollute;

(c) Each firm purifies with probability $1/(3 + \sqrt{3})$;

(d) Each firm purifies with probability $1/(3 - \sqrt{3})$.

4.23 Colonel Blotto Game

Colonel Blotto and the Folks' Militia each try to occupy two posts by properly distributing their forces. Colonel Blotto has 4 regiments and the Militia has 3 regiments. If Colonel Blotto has more regiments than the enemy at a post, Colonel Blotto receives the enemy's regiments plus one (i.e., one is value of occupying the post). If Colonel Blotto has fewer regiments at a post that the enemy he looses one plus the number of regiments he has at the post. A draw gives both sides zero. The total payoff is the sum of the payoff at the two posts.

Show that Colonel Blotto has 5 pure strategies and the Folks' militia has 4 pure strategies. Find the payoff matrix, and find a mixed strategy Nash equilibrium for the two sides.

4.24 Number Guessing Game

You pick a number from 1 to 3. I try to guess the number. You respond (truthfully!) by saying "high" "low" or "correct." The game continues until I guess correctly. You receive a number of dollars equal to the number of guesses I took.

(a) What are the our Nash strategies for this game?

(b) How much do you expect to win in this game?

4.25 Target Selection

There are n targets whose military values are $a_1, \ldots, a_n$, where $a_1 > a_2 > \ldots > a_n > 0$. The attacker has one attacking unit to allocate to one of the n targets, and the defender has one unit to allocate to the defense of the targets. If target k is attacked and is undefended, it will be captured, with the value a_k going to the attacker. If target k is defended, it has a probability p of being successfully held by the defender, so the expected payoff to the attacker is $(1-p)a_k$. Show that the unique Nash equilibrium has the following characteristics:

(a) There is some m such that only targets $1, \ldots, m$ are attacked and defended with positive probability;

(b) If x_i is the probability of attacking target i, then

$$a_i x_i = a_j x_j \qquad i, j \leq m.$$

(c) If y_i is the probability of defending target i, then

$$a_i(1 - py_i) = a_j(1 - py_j) \qquad i, j \leq m.$$

(d) If we define

$$A_k = \sum_{i=1}^{k} \frac{1}{a_i}$$

then the payoff of the game to the attacker is $(m-p)/A_m$, so m should be chosen to maximize this value.

4.26 A Reconnaissance Game

Attacker can either attack with all its forces (strategy 1), or attack with part of its forces, leaving the remainder as reserves and rear-guard in case its forces are outflanked (strategy 2). Defender has the same two strategy options. The payoff to attacker if attacker uses strategy i and defender uses strategy j is a_{ij}. We assume it is best for the attacker to use the same strategy as the defender; i.e., $a_{11} > a_{21}$ and $a_{22} > a_{12}$.

Attacker can also send out a reconnaissance force, at cost c, to find out defender's strategy. This will surely work, unless defender takes countermeasures at cost d (these countermeasures must be taken without knowing whether attacker will actually reconnoiter), in which case reconnaissance will fail.

(a) How many strategies does Defender have? How many does Attacker have?

(b) Show that elimination of dominant strategies can reduce the normal form matrix to 4×4.

(c) If the payoff matrix is

$$A = [a_{ij}] = \begin{bmatrix} 48 & 24 \\ 12 & 36 \end{bmatrix} \qquad \text{with } c = 9 \text{ and } d = 7,$$

find a Nash equilibrium in mixed strategies, assuming the payoff to the defender is the negative of the payoff to the attacker.

4.27 Attack on Hidden Object

A defender has an object of great value that he can carry in one of two identical bombers P (protected) and F (flank). The bombers fly in formation, so to attack P, one must fly past F and run the risk α, with $0 < \alpha < 1$, of being shot down before engaging P. Once the attacker has engaged his target (whether F or P), he can destroy it with probability β, with $0 < \beta < 1$. If F is intact, it is harder to destroy P than if F is gone. If F is intact, the probability of destroying P is $\gamma = (1 - \alpha)\beta$.

Suppose the attacker has two chances to hit a target and destroy the valued object. The defender has two strategies: locate the object in F and locate the object in P. The attacker has four strategies: attack F both times, attack P both times, attack F the first time and P the second time, and vice-versa. The understanding is that if the first attack was

successful, the second attack is directed against the remaining target, whatever the strategy used.

Determine the payoff matrix and find the Nash strategies.

4.28 Two-Person Zero-Sum Games

A *zero-sum game* is, appropriately enough, a game in which the sum of the payoffs to all the players is zero. Von Neumann and Morgenstern (1944), who launched modern game theory, lay great stress on zero-sum games, and indeed defined equilibrium in a way that only works for two-person zero-sum games (Nash had not yet invented the equilibrium concept that bears his name). Suppose the payoff to player 1 is $\pi(\sigma, \tau)$ when player 1 uses σ and player 2 uses τ, so the payoff to player 2 is $-\pi(\sigma, \tau)$. Von Neumann and Morgenstern defined (σ^*, τ^*) to be an equilibrium of the two-person zero-sum game if σ^* maximizes $\min_\tau \pi(\sigma, \tau)$ and τ^* minimizes $\max_\sigma \pi(\sigma, \tau)$. They showed that this *maximin solution* satisfies

$$\pi(\sigma^*, \tau^*) = \max_\sigma \min_\tau \pi(\sigma, \tau) = \min_\tau \max_\sigma \pi(\sigma, \tau). \tag{4.2}$$

It is easy to show that *a strategy profile in two-person zero-sum game is a Nash equilibrium if and only if it is a maximin solution.* This implies, in particular, that *all Nash equilibria of a two-person zero-sum game have the same payoffs.*

To prove a Nash equilibrium is a maximin solution, suppose (σ^*, τ^*) is a Nash equilibrium. Then we have

$$\begin{aligned} \max_\sigma \pi(\sigma, \tau^*) &\geq \min_\tau \max_\sigma \pi(\sigma, \tau) \geq \pi(\sigma^*, \tau^*) \\ &\geq \max_\sigma \min_\tau \pi(\sigma, \tau) \geq \min_\tau \pi(\sigma^*, \tau). \end{aligned}$$

The first inequality above is obvious, the second follows from

$$\min_\tau \max_\sigma \pi(\sigma, \tau) \geq \min_\tau \pi(\sigma^*, \tau) = \pi(\sigma^*, \tau^*),$$

the third follows from

$$\max_\sigma \min_\tau \pi(\sigma, \tau) \leq \max_\sigma \pi(\sigma, \tau^*) = \pi(\sigma^*, \tau^*),$$

and the fourth inequality is obvious. But the first and third terms must then be equal, since (σ^*, τ^*) is Nash, and similarly for the third and fifth terms. Thus they are all equal, so (σ^*, τ^*) is maximin.

To show that a maximin solution is a Nash equilibrium, suppose (σ^*, τ^*) is maximin. We know that the second equation in (4.2) holds

since there exists a Nash equilibrium, and we have already shown that a Nash equilibrium is maximin satisfying (4.2). But then we have

$$\pi(\sigma^*, \tau^*) \leq \max_{\sigma} \pi(\sigma, \tau^*) = \min_{\tau} \pi(\sigma^*, \tau) \leq \pi(\sigma^*, \tau^*).$$

This proves all three terms are equal, so (σ^*, τ^*) is maximin. Q.E.D.

4.29 An Introduction to Forward Induction

	U	D
L	$2, 2$	$3, 0$
R	$0, 3$	$4, 4$

The little game to the right has three Nash equilibria. The first is the Pareto-efficient equilibrium (D, R), and another is the less desirable (U, L). There is also a mixed strategy Nash equilibrium, as you can easily show. How do we choose among these equilibria? Which one is the 'right' one? Perhaps some natural *Nash equilibrium refinement* concept can be used to choose among them. We have found one such 'refinement,' that of *subgame perfection*, to be extremely insightful (see the Big Monkey and Little Monkey example 1.2), and we return to it in Chapter 5. Perhaps there are useful refinement concepts—in particular, one that can handle this little game.

Well, many game theorists believe that the Quest for the Correct Refinement must be unsuccessful, since which Nash equilibrium is appropriate (if any) depends on the ancillary informational conditions we expect to hold, and the specific dynamic governing the play of the game (more on this in Chapters 9 and 10). This does not mean, however, that certain 'plausible' refinements might not in fact select the same equilibria as certain 'plausible' dynamics under certain 'plausible' informational conditions. This little game gives us two interesting refinement criteria to think about: *risk dominance* and *forward induction*.

Notice that the (U, L) equilibrium has attractive property that each player does *better* if the other player in fact does *not* play the Nash strategy (receiving 3 as opposed to 2)! By contrast, in the (D, R) equilibrium each player does *much worse* if the other player chooses the 'wrong' strategy (receiving 0 as opposed to 4). Thus in a sense the $(2, 2)$ payoff is 'less risky' than the $(4, 4)$ payoff. To formalize this, we say a pair of strategies is *risk dominant* (Harsanyi and Selten 1988) if each strategy is a best response to a mixed strategy of the other player that weights all the player's pure strategies equally. The justification for this concept is the *Principle of Insufficient Reason* (§16.9.2), which says that

under conditions of complete ignorance concerning which of n outcomes will occur, you should treat the outcomes as equally likely. Armed with this definition, it is easy to show that (U, L) is indeed a risk dominant Nash equilibrium.

To illustrate forward induction, suppose player 2 has an 'outside option' that pays him 3, that he can take instead of playing the game. Then player one can reason as follows: "The only way player 2 can receive a payoff greater than 3 is if he plays R, and he believes that I will play D with greater than 75% probability. Thus I will play R."

(a) Find the mixed strategy Nash equilibrium of the 2×2 game above, and find the payoff to the two players in this equilibrium;

(b) Show that (U, L) is a risk dominant Nash equilibrium;

(c) Show that if we define another strategy, O (outside option), for player 2 that pays him 3, and pays player 1 some value a no matter what player one does, then there is a mixed strategy combination of R and O that strictly dominates L. But then D dominates U, and after eliminating T, R dominates O, so the iterated elimination of dominated strategies gives us the same result as forward induction.

4.30 Mutual Monitoring and Cooperation in a Partnership

	T	M
H	$\begin{matrix}0\\0\end{matrix}$	$\begin{matrix}0\\-c\end{matrix}$
S	$\begin{matrix}b\\-\lambda\end{matrix}$	$\begin{matrix}b-\alpha\\-\lambda(1-\alpha)-c\end{matrix}$

Two agents share a resource. One agent, whom we call the 'taker,' gets to use the resource, and can either steal (S) or be honest (H) in the amount of resource used. The other agent the 'watcher,' can monitor (M) or trust (T). We normalize the payoffs to the two players following the 'cooperative' strategy (T, H) to be zero. Let b be the benefit to the taker from stealing and not getting caught, let p be the loss to the taker from getting caught stealing, and let α be the probability of getting caught if the watcher monitors. Also, let c be the cost to the watcher of monitoring and let λ be the loss to the watcher if the taker steals and is not caught. We assume $b, p, \alpha, \lambda > 0$. We can normalize $b + p = 1$ (e.g., by dividing all of the payoffs to player 1 by $b + p$). The game matrix is then given in the above figure.

Let μ be the probability of monitoring in the watchers's mixed strategy, and let σ the probability of stealing in the taker's mixed strategy.

(a) Prove that if $c < \alpha\lambda$ and $b < \alpha$ then there is a completely mixed strategy Nash equilibrium with $\mu = b/\alpha$ and $\sigma = c/\alpha\lambda$. Show that the payoff to the taker is zero, and the payoff to the watcher is $-c/\alpha$.

(b) Explain why the loss to the watcher depends only on c and α, and not, for instance, on λ. Explain why the return to the taker doesn't depend on any of the parameters of the problem, so long as $c < \alpha\lambda$ and $b < \alpha$.

(c) What are the Nash equilibria if one or both of the inequalities $c < \alpha\lambda$ and $b < \alpha$ is violated?

4.31 Mutual Monitoring and Cooperation in Teams

This is a continuation of the previous problem. Now suppose there are $n+1$ agents, where agent $n+1$ is the 'taker,' agents $1, \ldots, n$ being identical 'watchers.' Suppose each watcher has the same probability $\alpha > 0$ of detecting stealing if monitoring, the same cost $c > 0$ of monitoring, and the same loss $\lambda > 0$ from an undetected theft. We only care about *symmetric equilibria*, in which all watchers choose the *same* probability μ of monitoring the taker.

Let $b < 1$ be the gain to the taker from stealing, and define

$$\rho = \alpha\lambda(1-b)^{\frac{n-1}{n}}.$$

Answer the following questions, assuming $b < \alpha\lambda$ and $c < \rho$.

(a) Show that there is a mixed strategy Nash equilibrium with the probability σ of stealing and the probability μ of monitoring given by

$$\sigma = \frac{c}{\rho} \qquad \mu = \frac{1-(1-b)^{\frac{1}{n}}}{\alpha}.$$

(b) Show that the payoff to a watcher is now $-c/\alpha(1-b)^{\frac{n-1}{n}}$. Why does this not depend on λ?

(c) How does this solution change as the group size n increases? Why doesn't the tragedy of the commons (i.e., the free rider) result hold in this case?

(d) What would happen as n increases if, for some fixed λ^*, we wrote $\lambda = \lambda^*/n$? This formulation would be reasonable if a dishonest taker imposed a fixed cost on the group no matter what its size, the cost being shared equally by the watchers.

4.32 Altruism(?) in Bird Flocks

This is an application of the results of the previous problem. Consider a flock of n birds eating in a group. A cat can catch a bird if it can sneak up behind a nearby rock without being seen. Each bird has an incentive to let the other birds look out for the cat while it conserves all its resources for eating (studies show that birds dissipate a considerable amount of energy and lose a considerable amount of time looking out for enemies). Why then do birds actually look out for predators when they eat in flocks? Are they 'altruistic?' Perhaps not.[3]

Suppose it takes the cat one second out in the open to reach the rock. If seen during that one second by even one bird, the birds will all fly off, and the cat will lose $p \in (0,1)$ in wasted time. If the cat reaches the rock, it catches one of the birds for a gain of $b = 1 - p$.

If each bird looks up from eating every $k \geq 1$ seconds, it will see the cat with probability $1/k$. Thus we can take $\alpha = 1$ and $\mu = 1/k$ in the previous problem (T corresponds to $k = \infty$). The cost to the bird of being caught is $\lambda = 1$, and the cost of looking up once is c. Prove the following, where we define $\mu = 1 - c^{1/(n-1)}$ and $\eta = 1 - c^{n/(n-1)}$.

Theorem. There are three types of symmetric equilibria.

(a) If $c > 1$ then no bird looks up, and the cat stalks the birds with probability one;

(b) If $c < 1$ and $b > \eta$, there is a symmetric Nash equilibrium in which the cat stalks with certainty and birds look up every $1/\mu$ seconds;

(c) If $b < \eta$, then there is a symmetric Nash equilibrium where the cat stalks with probability $\sigma = c(1-b)^{-(n-1)/n}$, and the birds look up with probability $1/\mu$.

[3] A similar problem arises in modeling the foraging behavior of flocks of birds in patchy environments, since if one bird finds a patch of food, all get to eat their fill (Motro 1991, Benkman 1988).

4.33 Robin Hood and Little John

Robin Hood and Little John both want to cross a rope bridge at the same time. There is only room for one. Each has two strategies go (G) and wait (W). It takes Robin Hood and Little John times τ_r and τ_{lj} respectively to cross the bridge. If both go at the same time, they fight it out at cost $\delta > 0$, after which the winner crosses the bridge. The probability of winning is 1/2 for each. If both wait, they play a polite little game of Alphonse and Gaston, at a cost $\epsilon > 0$, and one of them eventually crosses first, again with probability 1/2. We assume $0 < \epsilon < \delta$, while τ_r and τ_{lj} represent the cost of waiting for the other to cross the bridge.

(a) Find the payoff matrix for this game; consider each player's cost as not including the necessary crossing time for himself;

(b) Find all pure and mixed strategy equilibria, and show how the relative values of τ_{lj}, τ_r, and δ determine which holds;

(c) Show that the larger δ, the less likely a go-go situation emerges.

(d) What is a socially optimal δ? Show that if Robin Hood always waits, he would gain by an appropriate reduction in the costs of fighting, but would not gain by an increase in the costs of fighting. What is the social significance of this?

4.34 The Motorist's Dilemma

	G	W	C
G	$-\delta-\tau/2$ $-\delta-\tau/2$	0 $-\tau$	0 $-\tau$
W	$-\tau$ 0	$-\epsilon-\tau/2$ $-\epsilon-\tau/2$	$-\tau$ 0
C	$-\tau$ 0	0 $-\tau$	$-\delta-\tau/2$ $-\delta-\tau/2$

George and Martha, travelling in opposite directions, come to an intersection and each wants to turn left, so one must wait for the other. The time to turn left is the same for both, and is equal to $\tau > 0$. The cost of both waiting is $\epsilon > 0$, and of both going at the same time is $\delta > \epsilon > 0$. In addition to the two strategies G (go) and W (wait), we add a third, C (contingent). Playing C means choosing W if the other driver chooses G, and choosing G if the other driver chooses W. If both drivers choose C, we treat this as a foul-up as serious as GG. The payoff matrix is thus as in the figure. Assuming $\tau < 2\delta$, so GG is not a dominant strategy, find the Nash equilibria.

4.35 Family Politics

Male	Female: Coy	Female: Loose
Faithful	$v-r-w$ $v-r-w$	$v-r$ $v-r$
Philanderer	0 0	v $v-2r$

In a certain species of bird (actually, this is true of many species of bird) males are faithful or philanderers, females are coy or loose. Coy females insist on a long courtship before copulating, while loose females do not. Faithful males tolerate a long courtship and help rear their young, while philanderers do not wait and do not help rear. Suppose v is the value of having offspring to either a male or a female, $2r > 0$ is cost of rearing an offspring, and $w > 0$ the cost of prolonged courtship to both male and female. We assume $v > r + w$. This means that if courtship leads to sharing the costs of raising an offspring, then it is worth it to both birds. The normal form matrix is shown to the right.

(a) Show that if $v > 2r$ there is one Nash equilibrium with only loose females and only philandering males.

(b) Show that if $v < 2r$, there is a unique completely mixed strategy for males and females. The fraction q of females who are coy is then given by $q = r/(v - w)$ and the fraction p of males who are philanderers is given by $w/(2r + w - v)$.

4.36 Frankie and Johnny

Frankie must pay Johnny a certain amount of money as compensation for shooting Johnny's lover, but they disagree on the amount. They agree on a negotiator, who will pick whichever of Frankie's bid x_f and Johnny's bid x_j is closer to the negotiators opinion y. Frankie and Johnny don't know y, but they know it is drawn from a distribution F such that $\Pr\{y < \tilde{y}\} = F(\tilde{y})$. What is the nature of a Nash equilibrium for this game?

4.37 A Card Game

There are two players, each of whom bets \$1 and receives a number between zero and one (uniformly distributed). Each player observes only his number. Player 1 can either fold or raise \$5. Player 2 can either

fold or see. If neither folds, the player with the higher number wins. Describe a Nash equilibrium for this game.

4.38 Cheater-Inspector

There are n rounds in a game between an inspector and a 'taker.' The taker can cheat in any round, and the inspector can inspect in any round. If the taker cheats without getting inspected, the game stops and she gains 1 in that period, and every remaining period. If the taker is inspected while cheating, the game stops, and she is fined a in that period and in every remaining period. If the taker is honest, she receives b in that round from the inspector if inspected and nothing if not inspected, and the game goes on. The game is zero-sum (i.e., whatever the taker gets the inspector loses). What is the payoff of the game to the taker, and what is an optimal strategy for taker and inspector? Hint: Let g_n be the payoff of game of length $n > 0$, and let $g_0 = 0$. Then for any n we have the game in Figure 4.4. Solve it and find g_n recursively (i.e., starting with $n = 1$, then $n = 2$, and so on).

	trust	inspect
cheat	n	$-na$
honest	g_{n-1}	$b + g_{n-1}$

Figure 4.4: Cheater-Inspector

4.39 The Groucho Marx Game

Two players ante an amount $a \geq 0$, and cards numbered $1, \ldots, n$ are placed in a hat. Each player draws a card, observing his own but not the other's. They simultaneously and independently decide to stay (s) or raise (r) by betting an additional $b \geq 0$. The high card wins $a + b$, or if one player raises and the other stays, the one raising wins a.

(a) Show that when $n = 3$, if $a \geq b$ a Nash equilibrium involves staying if you pick a 1, and raising otherwise. If $a < b$, a Nash equilibrium is to stay unless you pick the 3. Hint: Show that staying if you pick the 3 is strictly dominated by raising. Then write 'rr' for "raise if you pick a 1, raise if you pick a 2", write 'rs' for "raise if you pick a 1, stay if you pick a 2," etc. This gives four strategies: {rr,rs,sr,ss}.

The normal form payoff matrix is thus 4×4. Calculate the entries by finding the expected return to each pair of strategies.

(b) Show that when $n = 4$ (a) if $b > a = 0$ it is Nash to stay unless you pick the 4; (b) if $2b > a > 0$, it is Nash to stay unless you get a 3 or a 4; (c) if $a > 2b > 0$, it is Nash to stay if you get a 1, and raise otherwise. Hint: eliminate strongly and weakly dominated strategies before forming the payoff matrix.

(c)* Using probability theory (especially conditional probabilities, as developed in §16.2), find the normal form of the Groucho Marx game for arbitrary $n > 0$. You may eliminate weakly dominated strategies, and hence assume the only pure strategies take the form of choosing a particular number, and raising only if your card is greater than that number.

(d)* Write the matrix for the normal form game for $n = 5$, and show that when $a = 1$ and $b = 2$,there is exactly one Nash equilibrium, given by

$$0.125s_1 + 0.375s_3 + 0.5s_4.$$

This is difficult only because there are lots of alternatives to check. I did this by writing a program that solves normal form games.

(e)* Write the matrix for the normal form game for $n = 6$, and show that when $a = 1$ and $b = 2$, there is exactly one Nash equilibrium, given by

$$0.083s_1 + 0.667s_4 + 0.25s_5.$$

This is even more tiresome to verify without the appropriate computer software.

(f)* Write the matrix for the normal form game for $n = 7$, and show that when $a = 1$ and $b = 2$, there is exactly one Nash equilibrium, given by

$$0.063s_1 + 0.937s_5.$$

4.40 Real Men Don't Eat Quiche

Clem is sitting at the best seat in the sports bar minding his own business when in walks the Dark Stranger (DS). The DS would like to Bully Clem to get the seat, but only if Clem is a Wimp. If Clem is Tough, DS prefers to Defer to Clem. DS doesn't know whether Clem is a Wimp or is Tough,

but he knows that with probability 1/3 Clem is Tough. If Clem is a Wimp and DS Bullies, Clem will move, but if Clem is Tough and DS Bullies, he will fight for his place at the bar. Clem gets 2 points if DS Defers to him, plus 1 point for avoiding consuming something he doesn't like. DS gets 1 point for guessing correctly.

Suppose Clem can signal whether he is Tough or a Wimp by choosing to consume either Beer or Quiche. Real men (Tough) don't like quiche and Wimps don't like beer. Find all the Nash equilibria to this game. The extensive form game, with payoffs, is shown in Figure 4.5

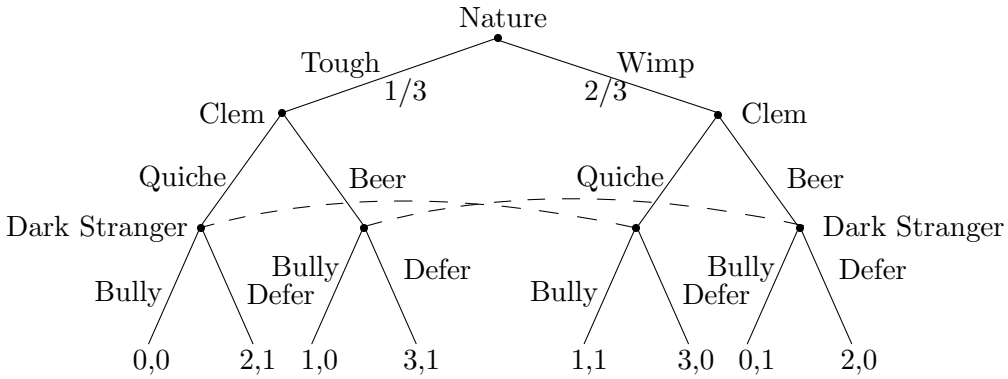


Figure 4.5: Real Men Don't Eat Quiche

4.41 The Vindication of the Hawk

Chicken is a two-player game in which each player can either attack (A) or remain peaceful (P). Suppose at the start of the game, each player has one util of Good Stuff. If both players remain peaceful, they each get to consume their Stuff. If one is peaceful and the other attacks, the attacker takes the other's Stuff. But if both attack, they each lose $a > 0$ utils. this gives us the normal form matrix to the right.

	A	W
A	$-a, -a$	0,2
W	2,0	1,1

(a) Show that there is a unique mixed strategy equilibrium, and the payoff of the game to the players *increases* when the potential loss from conflict, a, increases.

(b) Show that if there is a fixed probability $p > 0$ that a player will attack by mistake (e.g., because the wrong button got pressed), this result no longer holds.

(c) What happens if one or both of the players develops a 'detector' that can give an early warning if the other player attacks?

4.42 Correlated Equilibria

Consider the Up-Down/Left-Right game played by Alphonse and Gaston, with normal form matrix shown in the figure. There are two Pareto-efficient pure strategy equilibria: (1,5) and (5,1). There is also a symmetric mixed-strategy equilibrium with payoffs (2.5,2.5), in which Alphonse plays u with probability 0.5, and Gaston plays l with probability 0.5.

	l	r
u	5,1	0,0
d	4,4	1,5

If the players can jointly observe an event with probability 1/2, they can achieve the payoff (3,3) by playing (u, l) when the event occurs, and (d, r) when it does not. Note that this is Nash, since if the event occurs and Gaston plays l, Alphonse' best response is u; if the event does not occur and Gaston plays r, then Alphonse' best response is d; and similarly for Gaston. This is called a *correlated equilibrium*.

A more general correlated equilibrium for this coordination game can be constructed as follows: build a device that has three states: a,b, and c, with probability of occurrence α, β, and $1 - \alpha - \beta$. Allow Alphonse to have the information set $[\{a\}, \{b, c\}]$, and allow Gaston to have the information set $[\{a, b\}, \{c\}]$. For what values of α and β is the following Nash: Alphonse plays u when he sees a and plays d when he sees $\{b, c\}$; Gaston plays r when he sees c and plays l when he sees $\{a, b\}$.

Notice that when a occurs, Alphonse sees a, so he knows that Gaston sees $\{a, b\}$, so Gaston plays l. Thus Alphonse' best response is u. So far so good. When b occurs, Alphonse sees $\{b, c\}$, so using Bayes' Rule, he knows that Gaston sees b with probability $\beta/(1-\alpha)$, and Gaston sees c with probability $(1-\alpha-\beta)/(1-\alpha)$. Thus Alphonse knows he faces the mixed strategy l played with probability $\beta/(1-\alpha)$ and r played with probability $(1-\alpha-\beta)/(1-\alpha)$. The payoff to u in this case is $5\beta/(1-\alpha)$ and the payoff to d is $4\beta/(1-\alpha) + (1-\alpha-\beta)/(1-\alpha)$. If d is to be a best response, we must thus have $1 \geq \alpha + 2\beta$. If c occurs, the same conditions for Alphonse hold.

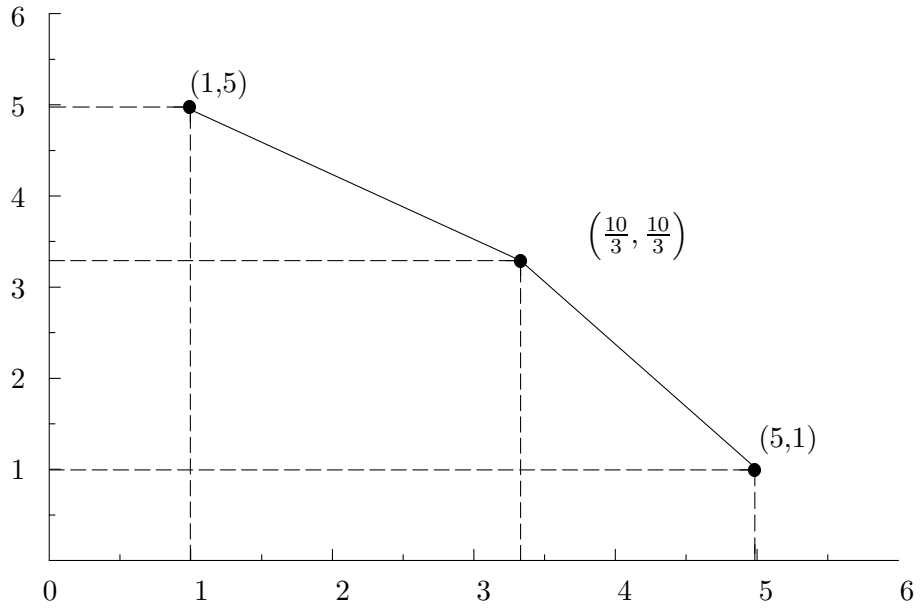
What about the conditions for Gaston? When c occurs, Alphonse sees $\{b, c\}$, so he plays d. Gaston's best response is r. So far so good. When a occurs, Gaston sees $\{a, b\}$, so his Bayesian posterior for the probability that Alphonse sees a is then $\alpha/(\alpha + \beta)$. A straightforward argument, parallel to that of the previous paragraph, shows that playing l is a best response if and only if $\alpha \geq \beta$.

Any α and β that satisfy $1 \geq \alpha + 2\beta$ and $\alpha \geq \beta$ permit a correlated equilibrium. Another characterization is $\beta \leq 1/3$ and $1 - 2\beta \geq \alpha \geq \beta$.

What are the Pareto optimal choices of α and β? Since the equilibrium is $a \to (u, l)$, $b \to (d, l)$, and $c \to (d, r)$, the payoffs to (a, b, c) are

$$\alpha(5, 1) + \beta(4, 4) + (1 - \alpha - \beta) = (1 + 4\alpha + 3\beta, 5 - 4\alpha - \beta),$$

where $\beta \leq 1/3$ and $1 - 2\beta \geq \alpha \geq \beta$. This is a linear programming problem. The solution is show below:



The pair of straight lines connecting (1,5) to (10/3,10/3) to (5,1) is the set of Pareto optimal points. Note that the symmetric point (10/3,10/3) corresponds to $\alpha = \beta = 1/3$.

4.43 Poker with Bluffing Revisited

If you have access to computer software to solve for Nash equilibria of normal form games, you can easily do the following. Doing the problem by hand is not feasible.

(a)* Show that there are two Nash equilibria to Poker with Bluffing (§2.17). Ollie uses the same strategy in both, bluffing on the first round with 40% probability (i.e., he raises with H or L on the first round, and drops with M). Stan has two mixed strategy best responses to Ollie, one of which uses two pure strategies and the other uses three. The latter involves bluffing with 40% probability. The expected payoff to the game for Ollie is \$1.20 (it's the same for both equilibria, since it's a zero sum game—see 4.27).

(b)* Now suppose Stan sees Ollie's first move before raising or staying. Show that there are 21 Nash equilibria, but that Stand uses only two different mixed strategies, and both involve raising with High and raising with Medium or Low with 25% probability on the first round, and calling with High or Medium and calling with 25% probability on the second round. Stan has lots of mixed strategies, but they entail only two different behaviors at the nodes where he chooses. If Ollie raised, Stan raises with H, raises with 75% probability with Medium, and stays with Low. If Ollie stayed, Stan raises with H, and raises with 25% probability with Low. In one set of strategies, Stan raises with 50% probability with Medium, and with 25% probability in the other. In all Nash equilibria Ollie can expect to lose \$0.25.

4.44 The Equivalence of Behavioral and Mixed Strategies

How do we get from behavioral to mixed strategies? A behavioral strategy p_i for player i has a *mixed strategy representation* σ_i defined in a notationally challenged but conceptually straightforward manner. Let $\mathcal{N}_i$ be the information sets for player i, and for $\nu \in \mathcal{N}_i$, let $a^\nu = \{a_1^\nu, \ldots, a_{k_\nu}^\nu\}$ be the actions available to player i at information set ν. Let S_i be the set of pure strategies for player i. Then $s \in S_i$ is the choice of a particular action $s(\nu) = a \in a^\nu$ for each $\nu \in \mathcal{N}_i$. Using p, the probability α_s^p that the player uses pure strategy s is then defined to be

$$\alpha_s^p = \prod_{\nu \in \mathcal{N}_i} p(s(\nu)).$$

Thus the mixed strategy representation σ_i of the behavioral strategy p_i is given by

$$\sigma_i = \sum_{s \in S_i} \alpha_s^p s.$$

You can check that σ_i is in fact a mixed strategy, i.e., $\sum_{s \in S_i} \alpha_s^p = 1$. Moreover, assuming perfect recall (§1.2), if p is a behavioral strategy profile with mixed strategy representation σ, then the payoffs to p and to σ are the same (see below).

Starting from a mixed strategy σ of the normal form game, there is also a straightforward, though again notationally challenged, way of deriving a behavioral strategy representation of σ. Let $\sigma_i = \sum_{s \in S_i} \alpha_s s$ be a mixed strategy for player i, and let $\nu \in \mathcal{N}_i$ with actions $a^\nu = \{a_1^\nu, \ldots, a_{k_\nu}^\nu\}$. Let $S_i[\nu] \subseteq S_i$ be the pure strategies that reach ν with positive probability for some strategy profile of the other players, and define $\mathrm{P}[\nu|\sigma_i] = \sum_{s \in S_i[\nu]} \alpha_s$. Thus $\mathrm{P}[\nu|\sigma_i]$ is the probability that i will choose a pure strategy that reaches ν with positive probability from some choices of the other players. Now for each $a \in a^\nu$, let $S_i[a] \subseteq S_i[\nu]$ be the pure strategies in $S_i[\nu]$ that choose a at ν, and define $\mathrm{P}[a|\sigma_i] = \sum_{s \in S_i[a]} \alpha_s$. Finally, let

$$p_i(a) = \frac{\mathrm{P}[a|\sigma_i]}{\mathrm{P}[\nu|\sigma_i]},$$

provided the denominator is not zero. If the denominator is zero, define $p_i(a)$ arbitrarily, just so $\sum_{a \in a^\nu} p_i(a) = 1$. We call p_i a *behavioral strategy representation* of σ_i. Notice that p_i is uniquely defined only if all information sets are attained with positive probability.

If σ is the mixed strategy representation of behavioral strategy p, then p is a behavioral representation of σ (see below). However there is no unique correspondence between behavioral and mixed strategies. This is because many mixed strategies can have the *same* behavioral strategy representation. For instance in One-Card Two-Round Bluff (4.11), the pure strategies $rrff$ and $rrfr$ have the same behavioral strategy representation, because if you fold on the first round, it does not matter whether you fold or raise on the second round, since you never get there! Notice that in fact in our description of the game, without comment we identified the two strategies with the single strategy rrf.

For another example of multiple mixed strategies representations of the same behavioral strategy, consider the Big Monkey and Little Monkey game tree in §1.1, where Little Monkey's mixed strategy set is

$$\{\alpha \mathrm{ww} + \beta \mathrm{pw} + \gamma \mathrm{wp} + \delta \mathrm{pp} \,|\, \alpha, \beta, \gamma, \delta \geq 0,\ \alpha + \beta + \gamma + \delta = 1\}.$$

A typical behavioral strategy p for Little Monkey is to play w with probability μ_w if Big Monkey plays w, and play w with probability μ_p if Big Monkey plays p. For concreteness, let us assume that $\mu_w + \mu_p \leq 1$ and $\mu_w \leq \mu_p$. You can check that $\mu_p \text{pw} + \mu_w \text{wp} + (1 - \mu_w - \mu_p)\text{pp}$ is a mixed strategy representation of p. But another mixed strategy representation of p $\mu_w \text{ww} + (\mu_p - \mu_w)\text{pw} + (1 - \mu_p)\text{pp}$, and any convex combination of these two mixed strategies *also* gives rise to the same behavioral strategy p.

If this maze of symbols confuses you, try it out on the One-Card Two-Round Bluff game. You should get the same result as we did a few paragraphs earlier.

We say a behavioral strategy profile $p = (p_1, \ldots, p_n)$ is a *Nash equilibrium in behavioral strategies* if for each player i, p_i is a best response to p_{-i}. Since behavioral strategies have the same payoffs as their mixed strategy representations (assuming perfect recall, which we do), p_i is a best response to p_{-i} if and only if the mixed strategy σ_i representation of p_i is a best response to σ_{-i}, the mixed strategy representation of p_{-i}. We conclude that *a behavioral strategy is a Nash equilibrium of the extensive form game $\mathcal{G}$ if and only if the mixed strategy representation of p is a Nash equilibrium of the normal form of $\mathcal{G}$.*

This exercise asks you to prove some of the assertions made above. Assuming perfect recall, show that

(a) if p is a behavioral strategy profile for an extensive form game $\mathcal{G}$, and σ is the mixed strategy representation of p, then for any player i, σ_i is a mixed strategy, i.e., $\sum_{s \in S_i} \alpha_s^p = 1$;

(b)* if σ is the mixed strategy representation of behavioral strategy p, then p is a behavioral strategy representation of σ;

(c)* if p is a behavioral strategy profile with mixed strategy representation σ, then the payoffs to p and to σ are the same (Kuhn 1953);

(d) if σ is a Nash equilibrium of the normal form representation of the extensive game $\mathcal{G}$, and if p is a behavioral strategy representation of σ, then at every information set ν, p_ν is a best response to $p_{-\nu}$.